MATH 2050 C Lecture 1 (Jan 11, 2022)

Pre-requisites: MATH 1050/1058 (and MATH 1010/1018)

- Set theoretic concepts ($\forall, \exists, \in, \subseteq, \cap, \cup$)
- Number systems ($\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$)
- Functions $f: A \to B$ [Ref: Chap. 1]
- ★ Proof Writing

Thm: $\nexists\, r \in \mathbb{Q}$ s.t. $r^2 = 2$. [i.e. $\sqrt{2}$ is irrational.]

Proof: We will prove "by contradiction".

Suppose NOT. Then, $\exists\, r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Since $r \in \mathbb{Q}$, we can find $p, q \in \mathbb{Z}$, $q \neq 0$ s.t.

$$r = \frac{p}{q} \qquad \text{where } p, q \text{ are "relatively prime".}$$

- As $2 = r^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$ (#)

  i.e. p^2 is even $\Rightarrow$ p is even, i.e. $\exists\, k \in \mathbb{Z}$ s.t. $p = 2k$.

- Plug $p = 2k$ into (#).

$$4k^2 = p^2 = 2q^2 \Rightarrow q^2 = 2k^2 \qquad (\#\#)$$

Similar argument $\Rightarrow$ q^2 is even $\Rightarrow$ q is even

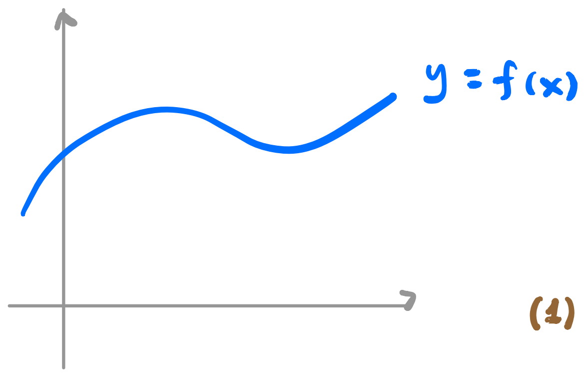
Thus, both p & q are even, which contradicts the fact that they are relatively prime.

$\square$

An Overview of MATH 2050 (and 2060 / 3060)

Goal: Study the "analytic properties" of functions $f: \mathbb{R} \to \mathbb{R}$

[e.g. $\underbrace{\text{limit, continuity}}_{\text{MATH 2050}}$, $\underbrace{\text{differentiable / integrable}}_{\text{MATH 2060 (3060)}}$?]



Q: $\exists f: \mathbb{R} \to \mathbb{R}$ s.t. continuous everywhere but nowhere differentiable ?

Summary (MATH 2050)

(1) [Ch. 2] $\mathbb{R}$ as complete ordered field.

(2) [Ch. 3] limit of sequences $\lim (x_n)$

(3) [Ch. 4] limit of functions $\lim\limits_{x \to a} f(x)$

(4) [Ch. 5] continuity of functions

Chapter 2 The Real Numbers

Grand Thm: $\mathbb{R}$ is a complete ordered field.

complete — analysis (topology)

ordered — inequalities

field — algebra

Field Properties

Def$^{\underline{n}}$/Thm: $(\mathbb{R}, +, \cdot)$ is a field, i.e.

$\exists$ two operations $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t. the following properties hold:

$+$

(A1) $a + b = b + a \quad \forall a, b \in \mathbb{R}$

(A2) $(a+b)+c = a+(b+c) \quad \forall a, b, c \in \mathbb{R}$

(A3) $\exists\, 0 \in \mathbb{R}$ s.t. $0 + a = a = a + 0 \quad \forall a \in \mathbb{R}$

(A4) $\forall a \in \mathbb{R}$, $\exists -a \in \mathbb{R}$ s.t. $a + (-a) = 0 = (-a) + a$.

$\cdot$

(M1) $a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in \mathbb{R}$

(M3) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $1 \cdot a = a = a \cdot 1 \quad \forall a \in \mathbb{R}$

(M4) $\forall a \in \mathbb{R}$, $a \neq 0$, $\exists \frac{1}{a} \in \mathbb{R}$ s.t. $\frac{1}{a} \cdot a = 1 = a \cdot \frac{1}{a} \quad \forall a \in \mathbb{R}$

$+$ $\cdot$

(D) $a \cdot (b+c) = a \cdot b + a \cdot c \quad \forall a, b, c \in \mathbb{R}$

$(b+c) \cdot a = b \cdot a + c \cdot a \quad \forall a, b, c \in \mathbb{R}$

Note: The remaining algebraic properties can be deduced from the field properties above.

Define: $a - b := a + (-b)$, $\frac{a}{b} := a \cdot \left(\frac{1}{b}\right)$

Notation: $a^n := \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$; $a^0 := 1$; $a^{-1} := \frac{1}{a}$

$n \in \mathbb{N}$

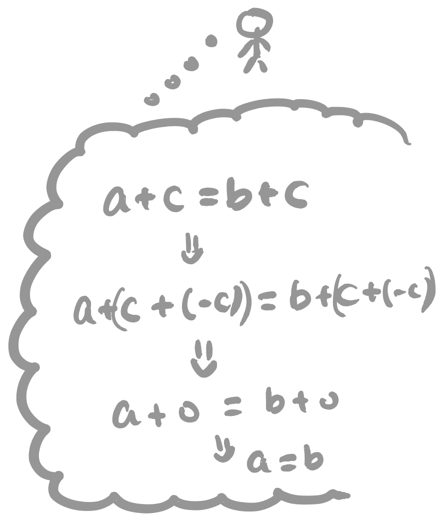
Prop: "Cancellation Laws"

(1) $a + c = b + c \Rightarrow a = b$

(2) $ac = bc$, $c \neq 0 \Rightarrow a = b$

Proof: (1)

$$
\begin{aligned}
a &= a + 0 && \text{(by (A3))} \\
&= a + (c + (-c)) && \text{(by (A4))} \\
&= (a + c) + (-c) && \text{(by (A2))} \\
&= (b + c) + (-c) && \text{(by assumption)} \\
&= b + (c + (-c)) && \text{(by (A2))} \\
&= b + 0 && \text{(by (A4))} \\
&= b && \text{(by (A3))}
\end{aligned}
$$



(2): Exercise.

$\square$

Cor: The zero element 0 in (A3) is unique.

Proof: Suppose there are two zero elements 0, $0'$. Then

$$0 \underset{0' \text{ (A3)}}{=} 0 + 0' \underset{0 \text{ (A3)}}{=} 0' \qquad \text{i.e. } 0 = 0'$$

$\square$

Exercise: 1 in (M3) is unique.

Prop: (1) $0 \cdot a = 0 \quad \forall a \in \mathbb{R}$

(2) $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$ (or both)

(3) $(-1) \cdot a = -a \quad \forall a \in \mathbb{R}$

Proof: (1) Consider

$$\cancel{0 \cdot a} + 0 \cdot a \overset{(D)}{=} (0+0) \cdot a \overset{(A3)}{=} 0 \cdot a \overset{(A3)}{=} \cancel{0 \cdot a} + 0$$

then by cancellation law (1), we have $0 \cdot a = 0$.

(2) Suppose $a \cdot b = 0$.

Case i: $a = 0 \Rightarrow$ Done.

Case ii: $a \neq 0$ [Want: Prove $b = 0$.]

Since $a \neq 0$, the inverse $\frac{1}{a} \in \mathbb{R}$ exists.

$$\cancel{a} \cdot b \underset{\text{by assumption}}{=} 0 \underset{\text{by (1)}}{=} \cancel{a} \cdot 0$$

By cancellation law (2), we have $b = 0$.

(3) Want to show: $a + (-1) \cdot a = 0$

Then, result follows from uniqueness of additive inverse "$-a$".

$$\begin{aligned} a + (-1) \cdot a &\overset{(M3)}{=} 1 \cdot a + (-1) \cdot a \\ &\overset{(D)}{=} (1 + (-1)) \cdot a \\ &\overset{(A4)}{=} 0 \cdot a \\ &\overset{\text{by (1)}}{=} 0 \end{aligned}$$

■

Remark: Other e.g. of fields $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_p$, $\left\{ \frac{\text{polynomials}}{\text{polynomials}} \right\}$ ......